University of Colorado
Department of Computer Science

Chaotic Dynamics – CSCI 4446/5446
Spring 2023

Problem Set 4

Issued: 7 February 2023

Due: 14 February 2023

Reading: *Strogatz, sections 2.0-2.3, 2.8, 6.0-6.5, 6.7 and chapter 5; sections 1 and 2 of Liz's ODE Notes;* Parker&Chua, chapter 4; Lorenz's "Computational Chaos..." article (listed below). Section 2 of Liz's TSA Notes may also be useful; see the course webpage. You can download a pdf of the Parker & Chua book from the link on the course webpage.

Online assignment: Tuesday: unit 3.2-3.4, 4.1, and 4.3-4.4 videos. Thursday: unit 7.4 and 5.2-5.4 videos. Friday: quizzes 3.2-3.4, 4.1, 4.3, and 5.2-5.3. **The video load this week is high. It'll back way off after this.**

Bibliography:

- D. Auerbach, P. Cvitanovic, J.-P. Eckmann, G. Gunaratne and I. Procaccia, "Exploring chaotic motion through periodic orbits," *Phys. Rev. Lett.* **58**:2387-2389 (1987). There's another UPO paper by Cvitanovic in the Campbell reprint collection, but it's heavy going.
- E. Lorenz, "Computational Chaos – A Prelude to Computational Instability," *Physica D*, **35**:299-317 (1989).
- J. Marsden *et al.*, "Symmetry, Stability, Geometric Phases, and Mechanical Integration," *Nonlinear Science Today*, volume 1, 1991.
- R. H. Miller, "A Horror Story about Integration Methods," *J. Computational Physics* **93**:469-476 (1991).
- W. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing*, Cambridge, 1988.
- D. Tritton, "Chaos in the Swing of a Pendulum," *New Scientist*, 7/24/86.
- H. C. Yee *et al.*, "Dynamical approach study of spurious steady-state numerical solutions of nonlinear differential equations. 1. The dynamics of time discretization and its implications for algorithm development in computational fluid dynamics," *Journal of Computational Physics* **97**:249-310 (1991)

Problems:

1. Write a fourth-order fixed-time-step Runge-Kutta integrator *from scratch*. Inputs should consist of a starting time t_0, time step Δt, number of steps n, and starting value $\vec{x}(t_0)$ for the state vector. If your language allows functions as arguments, make the system derivative an argument as well. The output of this procedure — a series of state vectors representing the n-point state-space trajectory emanating from $\vec{x}(t_0)$ — should go to a file.

Please do not use any canned numerical integration routines, commands, functions, etc. Later problem sets will use this integrator with many different systems; it will save you a **lot** of time later if you write this version to take arbitrary-size state vectors!

The rest of the problems in this set concern the following equation:

$$ml\frac{d^2}{dt^2}\theta(t) + \beta l\frac{d}{dt}\theta(t) + mg\sin\theta(t) = A\cos(\alpha t)$$

This is the equation of motion of a forced, damped pendulum. The state vector is $[\theta, \omega]^T$; the former is measured in radians and the latter in radians per second. Generate your solutions of this equation using your RK4 solver. There is no need to turn anything in for this problem.

2. Use m=0.1kg, l=0.1m, β=0 and set the drive amplitude and frequency to zero ($\alpha = A = 0$).

(a) Turn in a plot of the state-space trajectory emanating from the point $[\theta, \omega] = [3, 0.1]$ with $\Delta t = 0.005$. Is this initial condition near a fixed point? Which one? Is that point stable or unstable?

(b) Turn in a plot of the state-space trajectory emanating from the point $[\theta, \omega] = [0.01, 0]$. You'll have to use different coordinate axes from those in part (a) to get a good plot. Does this trajectory look more like a perfect ellipse than the trajectory of part (a)? *CSCI 5446 students: why or why not?*

3. Use your integrator to generate a *state-space portrait* of the system, using the coefficient values given in problem 2 above. This will entail selecting a representative set of starting points—i.e., a set of initial conditions set that samples all of the salient and interesting features of the dynamics. Simply dropping points on a grid is a bad idea here because you will oversample the boring areas and/or undersample the interesting ones. If you have trouble selecting a good set of points, look at figure 6.7.3 in Strogatz for an example. Turn in a copy of your plot.

4. Now repeat problem 3 with $\beta = 0.25$. What happens to the various features of the plot? What does this imply about the physical dynamics? Turn in a copy of this plot. What do you think would happen with a higher β? What about a lower β?

5. Modify your code so that it plots θ modulo 2π and see what that does to your results in problem 4. (Here are some examples that should help you understand what "modulo" means: 2 modulo 3 = 2; 3 modulo 3 = 0; 4 modulo 3 = 1; 7 modulo 3 = 1.) Make sure you understand this, and turn in a copy of the plot. If you have weird horizontal lines across your plot, it's probably connecting the dots. It shouldn't.

6. Leaving β at 0.25, turn on the drive. Vary the drive frequency α and amplitude A (i.e., "explore the parameter space") and describe and explain what you see on the plots — in the vocabulary of nonlinear dynamics, not physics (e.g., "bifurcation," etc.) Find a chaotic trajectory and turn in a plot of it.

Hints:

- Start with the drive frequency at about 3/4 of the natural frequency of the device and slowly increase the drive amplitude. (The natural frequency is related to g and l in the manner derived during the first week of the semester.)
- If you're having trouble diagnosing chaos, remember the "there's a pattern but it never quite repeats" heuristic.
- Make sure you plot θ modulo 2π!

7. Turn the drive back off, set β back to 0, play with the timestep, and describe the effects on the state-space portrait (Hint: try increasing the timestep until weird things happen. Describe and attempt to explain them.)