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University of Colorado
Department of Computer Science

Chaotic Dynamics – CSCI 4446/5446
Spring 2023

Problem Set 5

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Issued:	14 February 2023
Due:	21 February 2023

Reading: *Strogatz, chapter 9 and sections 4.4-4.6, 6.7, 10.6, 12.3, and 12.4; section 3 of Liz's TSA Notes.*

Parts of sections 4.5, 4.6, 9.1 and 9.2 in Strogatz are mathematically quite dense. Read for content here; *look* at the equations, but don't worry about understanding the derivations or the deep mathematical implications.

Note: this week's reading assignment is substantial and the programming workload has been adjusted accordingly.

Online assignment: Tuesday: unit 6.1-6.3 videos. Thursday: unit 7.3-7.4 videos. Friday: quizzes 6.1-6.3 and 7.3-7.4.
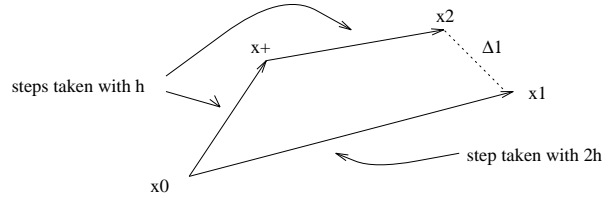
Bibliography:

- K. M. Cuomo and A. V. Oppenheim, "Circuit Implementation of Synchronized Chaos with Applications to Communications," *Physical Review Letters* **71**:65-68 (1993). (Strogatz discusses this in section 9.6.)

- D. D'Humieres *et al.*, "Chaotic States and Routes to Chaos in the Forced Pendulum," *Phys. Rev. A* **26**:3483, 1982.

- I. Epstein, "Nonlinear oscillations in chemical and biological systems," *Physica D* **51**:152-160 (1991).

- B.-L. Hao, *Chaos*, Singapore: World Scientific 1984. A nice reprint collection.

- E. Lorenz, "Deterministic Nonperiodic Flow," *J. Atm. Sciences* **20**:13, 1963.

- L. M. Pecora and T. L. Carroll, "Synchronization in Chaotic Systems," *Physical Review Letters* **64**:821-824 (1990). Strogatz discusses this in section 9.6.

- Ross, "Lorenz's Butterfly," *Scientific American*, September 1990.

- O. Rössler, "An Equation for Continuous Chaos," *Phys. Lett.* **57A**:397, 1976.

Problems:

0. Play with some of the tones at `tones.wolfram.com` and think about how they were generated. (No need to turn anything in here, but this will come back as an in-class thought experiment in a week or two.)

1. Write an *adaptive* time-step fourth-order Runge-Kutta integrator. A picture with definition of terms appears below:

$$\vec{\Delta_1} = \vec{x_2} - \vec{x_1}. \hspace{4cm} \text{Error condition: } ||\Delta_1||_\infty \leq 0.001.$$

Note: $||1, -2, -1||_\infty = 2$ and $||1, -2, -1||_2 = \sqrt{1^2 + (-2)^2 + (-1)^2}$. The former is the "largest element" operator and the latter is Euclidean length.

(No need to turn anything in here either; the following problems will put this beast through its paces.)

2. (a) Use your adaptive integrator to plot a picture of the chaotic attractor in the Lorenz system:

$$\vec{F}(\vec{x}, a, r, b) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a(y - x) \\ rx - y - xz \\ xy - bz \end{bmatrix} \hspace{2cm} (1)$$

with $a = 16$, $r = 45$, $b = 4$, starting from the initial condition $(x, y, z) = (-13, -12, 52)$. Unless you have a 3D plotter, you will have to choose a projection to make the plot; most books (e.g., Gleick) use the $x - z$ plane.

(b) Use your *nonadaptive* integrator from PS4 to plot a trajectory from the same initial condition using a timestep of 0.001. Overlay your plots from (a) and (b)—e.g., with different color points. Do the two solutions (adaptive and non) agree? (It may help to do this with only a short segment of the trajectory, so you can see what's going on.) Are the points produced by the adaptive solver evenly spaced in time?

(c) What happens if you keep a and b fixed and change r? (no need to turn in plots; simply describe what happens *in the language of nonlinear dynamics*). Hints: try a few r values between 0 and 1, several between 13.5 and 14, and a bunch between 23 and 30. Also, don't confine your explorations to a small area of the state space; try lots of different initial conditions.

3. Use your adaptive integrator to plot a picture of the chaotic attractor in the Rössler system:

$$\vec{F}(\vec{x}, a, b, c) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -(y + z) \\ x + ay \\ b + z(x - c) \end{bmatrix} \hspace{2cm} (2)$$

with $a = .398$, $b = 2$, $c = 4$. Again, choose any projection you wish.

4. Now go back to the Lorenz system and the parameter values given in problem 2. Your task in this problem is to explore the effects of loosening the error bounds. Increase the allowed error from 0.01 until you see the dynamics break down. Turn in one or two plots in this series, the last clearly showing the altered dynamics. How is this related to the results of the timestep explorations you did on the last problem set?